COMPLEX MULTIPLICATION: LECTURE

1. Weierstrass $\wp$ function

To motivate the following construction, let us consider again the situation over $\mathbb{Q}$. As was mentioned before, algebraic properties interact badly with holomorphic maps, so there is no reason that the image of an arbitrary point under a holomorphic map such as exp should be algebraic. The property of the point $2\pi im/n$ which guarantee that their images under exp is algebraic is that they are torsion points of group structure $\mathbb{C}/2\pi i\mathbb{Z}$, let us briefly explain why this is so.

The identity

$$\exp(a+b) = \exp(a)\exp(b)$$

shows that exp converts the group structure on $\mathbb{C}/2\pi i$ to multiplication on $\mathbb{C}^\times$. Multiplication on $\mathbb{C}^\times$ is an algebraic map, i.e. can be defined by polynomials, so that multiplication by n on $\mathbb{C}/2\pi i\mathbb{Z}$ is converted to the algebraic map $z \mapsto z^n$ on $\mathbb{C}^\times$. The polynomial X^n has coefficients in $\mathbb{Q}$ so that the roots of $X^n - 1 = 0$ must be algebraic numbers.

The important property that we are using is that exp converts the group structure to an algebraic map such that the "N-multiplication formula" is defined by polynomials with coefficients in $\mathbb{Q}$. Thus if we are trying to generalise this to elliptic curves, we should look for holomorphic maps which give us some algebraic interpretation of the complex torus and its group structure. Unfortunately since a complex torus is a compact Riemann surface, any holomorphic to $\mathbb{C}$ is constant. Thus we instead relax the condition of holomorphicity to meromorphicity. We will see that there is a very simple description of such maps.

We begin with the more general definition.

Definition 1.1. Let Λ be a lattice of $\mathbb{C}$. A meromorphic function f on $\mathbb{C}$ is said to elliptic with respect to λ if for all $\omega \in \Lambda$, we have

$$f(z+\omega) = f(\omega)$$

Remark 1.2. In the setting of Riemann surfaces, an equivalent definition of elliptic function is to give a holomorphic map of Riemann surfaces $\mathbb{C}/\Lambda \to \mathbb{P}^1(\mathbb{C})$.

We denote by A_Λ the set of elliptic functions with respect to Λ. The main theorem that we will try to prove in this lecture is the following.

Theorem 1.3. *Fix a lattice Λ_τ, there exists an elliptic function $\wp$ with a double pole at $\omega \in \Lambda_\tau$ such that:*

i) There is a bijecction

$$\mathbb{C}/\Lambda - (0,0) \ni z \leftrightarrow (\wp, \wp')$$

where the righthand is the subset of $\mathbb{C}^2$ which satisfy $Y^2 = 4X^3 - g_2(\tau)X - g_3(\tau)$

ii) $A_\Lambda = \mathbb{C}[\wp, \wp']$

where $g_2(\tau)$ and $g_3(\tau)$ are the functions defined in the last section

The function constructed is known as the Weierstrass $\wp$ function. The second part of the theorem shows in some in some sense, $\wp$ is the most basic elliptic function in that any other function can be written as a polynomial in $\wp$ and its derivative.

For the rest of this section, we fix a lattice $\Lambda = \langle 1, \tau \rangle$.

Definition 1.4. Define the Weierstrass $\wp$ function with respect to Λ to be the function given by the infinite series.

$$\wp(z) = \frac{1}{z} + \sum_{\omega \in \Lambda - (0,0)} \frac{1}{(z-\omega)^2} - \frac{1}{w^2}$$

We must check this is well-defined.

Proposition 1.5. *The above infinite series converges to an elliptic function which is holomorphic outside* Λ

Proof. The idea of the proof is the same as proof for Eisenstein series. Let $|\omega| > 2|z|$ and suppose $z \notin \Lambda$, then

$$|\frac{1}{(z-\omega)^2} - \frac{1}{w^2}| = |\frac{\omega^2 - (z-\omega)^2}{(z-\omega)^2\omega^2}|$$
$$= |\frac{2z\omega}{(z-\omega)^2\omega^2}|$$
$$< |\frac{10z}{\omega^3}|$$

Thus there are constant A and B such that

$$\sum_{\omega \in \Lambda - (0,0)} |\frac{1}{(z-\omega)^2} - \frac{1}{w^2}| < A + \sum_{\omega \in \Lambda - 0, |\omega| > 2|z|} |\frac{10z}{\omega^3}|$$

It follows from the proof of Proposition 1.7 from Lecture 6 that as z ranges over compact subset of $\mathbb{C} \backslash \Lambda$ the series converges absolutely and uniformly. Hence

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda - (0,0)} \frac{1}{(z-\omega)^2} - \frac{1}{w^2}$$

defines a meromorphic function on $\mathbb{C}$.

It remains to show that $\wp$ is elliptic. To do this we compute $\wp'(z)$. Since the series defining $\wp$ converges absolutely, we can calculate $\wp'(z)$ by differentiating term by term. We obtain:

$$\wp'(z) = -\frac{2}{z^3} + \sum_{\omega \in \Lambda - (0,0)} \frac{-2}{(z-\omega)^3} = \sum_{\omega \in \Lambda} \frac{-2}{(z-\omega)^3}$$

Clearly this function is elliptic with respect to Λ. Define

$$\lambda(z) = \wp(z) - \wp(z+\tau)$$

Since $\wp'$ is elliptic, it follows that $\lambda'(z) = 0$ and so λ is a constant C say. But it is clear for the series definition of $\wp$ that $\wp(z) = -\wp(-z)$, hence

$$\wp(-\tau/2) = \wp(\tau/2) = \wp(-\tau/2 + \tau) = \wp(\tau/2) + C$$

Hence $C = 0$. Applying this with τ replaced by any element of Λ, we obtain $\wp$ is elliptic. $\square$

The function $\wp$ has the following properties

Proposition 1.6. *i)* $\wp$ *is holomorphic outside* Λ *and has a pole of order* 2 *at any* $\omega \in \Lambda$.

ii)

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)E_{k+1}(\tau)z^{2k}$$

where E_{k+1} *is the Eisenstein series defined in the last lecture.*

iii) $\wp'(z)^2 = 4\wp(z)^3 - g_2(\tau)\wp(z) - g_3(\tau)$

Proof. i) The holomorphicity outside Λ follows from the proof of the last proposition. $\wp(z)$ has a pole of order 2 at 0 hence since it is elliptic has a pole of order 2 at each $\omega \in \Lambda$.

ii)

$$\wp(z) - \frac{1}{z^2} = \sum_{\Lambda - \backslash 0} \frac{1}{\omega^2}\frac{1}{(1-\frac{z}{\omega})^2} - \frac{1}{\omega^2} = \sum_{\Lambda \backslash 0}\left(\frac{1}{\omega^2}\left(\sum_{n=0}^{\infty}\left(\frac{z}{\omega}\right)^n\right)^2 - \frac{1}{\omega^2}\right)$$

$$= \sum_{\Lambda \backslash 0}\left(\frac{1}{\omega^2}\sum_{n=0}^{\infty}(n+1)\left(\frac{z}{\omega}\right)^n - \frac{1}{\omega^2}\right) = \sum_{n=1}^{\infty}(n+1)\sum_{\Lambda - \{(0,0)\}}\frac{z^n}{\omega^{n+2}}$$

$$= \sum_{k=1}^{\infty}(2k+1)\sum_{\Lambda - \backslash 0}\frac{1}{\omega^{2k+2}}z^{2k} = \sum_{k=1}^{\infty}(2k+1)E_{k+1}(\tau)z^{2k}$$

iii) We have

$$\wp(z) = \frac{1}{z^2} + 3E_2(\tau)z^2 + 5E_3(\tau)z^4 + ...$$

$$\wp'(z) = \frac{-2}{z^3} + 6E_2(\tau)z + 20E_3(\tau)z^4$$

Thus calculating

$$\wp'(z)^2 - 4\wp(z)^3 + g_2(\tau)\wp(z) + g_3(\tau)$$

we find that the coefficients of z in negative degrees vanish hence this is a holmorphic function. Since it also also elliptic, it must be constant. Evaluating at $z = 0$ we see that this constant is 0. Hence the equality holds. □

It is slightly miraculaous that the mermorphic function satisfies this polynomial relationship with its derivative. We will now use these two functions to embed the elliptic curve E_Λ into $\mathbb{C}^2$.

Proposition 1.7. f *be a meromorphic function on* $\mathbb{C}$, *we let* $v_z(f)$ *denote the order of vanishing of* f *at* z. *Suppose* f *is elliptic with respect to the lattice* Λ. *Then*

$$\sum_{z \in \mathbb{C} \backslash \Lambda} v_z(f) = 0$$

Proof. Since $\mathbb{C}/\Lambda$ is compact, it follows by the isolation of zeros/poles theorem that there f has only finitely zeros or poles in $\mathbb{C}/\Lambda$ hence the sum is well defined. Taking any fundamental parallelogram whose edges do not contain any pole of zero of f, we have since f is elliptic

$$\int_\gamma \frac{f'}{f}dz = 0$$

where γ traces a path around this parallelogram. By the argument principle, this integral is precisely the sum in question. $\square$

It follows that within any fundamental paralleogram there are exactly 2 zeros of $\wp$ counted with multiplicity.

In fact we can say something a lot more precise than this.

Proposition 1.8. *for all $a \in \mathbb{C}$, there exists a $w \in \mathbb{C}$ such that $\wp(w) = a$. The set of such solution are $\{w, -w\}$ if $w \notin \{\frac{\omega}{2} | \omega \in \Lambda\}$.*

Proof. Consider the equation $\wp(z) - a$. This is an elliptic function with a double pole at 0, hence by the previous proposition we have that the sum of zeros of this function is 2 (with multiplicity). Let w be a zero, then if $w \notin \{\omega/2 | \omega \in \Lambda\}$, $-w$ is another zero and hence these are all the zeros of $\wp(z) - a$.

Suppose $w \in \{\omega/2 | \omega \in \Lambda\}$. Since $\wp$ is an even function, $\wp'$ is odd so that $\wp'(z) = -\wp'(z + \omega)$ for all $\omega \in \Lambda$. It follows that $\wp'(w) = 0$ hence wp has a double zero at w and hence w is the only zero. $\square$

Corollary 1.9. *The polynomial*

$$4X^3 - g_2(\tau)X - g_2(\tau)$$

has roots $\wp'(\frac{\tau}{2}), \wp'(\tau(\frac{1}{2})), \wp'(\frac{1+\tau}{2})$ and these are distinct.

Proof. Since $\wp'$ is odd and elliptic,

$$-\wp'(\frac{\tau}{2}) = \wp'(-\frac{\tau}{2}) = \wp'(\frac{\tau}{2})$$

Hence $\wp'(\frac{\tau}{2}) = 0$, and similarly for the other two points. These are all distinct since otherwise $\wp(z) - \wp(w)$ has 4 roots with multiplicity where w is one of $\frac{\tau}{2}, \frac{1}{2}, \frac{1+\tau}{2}$, hence they are all the roots of the polynomial. $\square$

We are now in a position to prove the main theorem.

Proof. By Proposition 1.6 iii) the map

$$\mathbb{C}/\Lambda - 0 \to \{(x, y) | y^2 = 4x^3 - g_2(\tau) - g_3(\tau)\}$$

given by $z \to (\wp(z), \wp'(z))$ is well defined.

For (x, y) in the set on the right, we know by the previous corollary that there is a $w \in \mathbb{C}/\Lambda$ such that $\wp(w) = x$. Since $\wp'(w)$ satisfies $\wp'(w)^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$, if $\wp'(\tau) \neq 0$, then $w \notin \{\omega/2 | \omega \in \Lambda\}$, and so $\wp'(w)$ and $\wp(-w)$ give the requisite two values of y. If $\wp'(w) = 0$, $w \notin \{\omega/2 | \omega \in \Lambda\}$, the is only one value of w for which $\wp(w) = x$ corresponding to the only one value of y.

ii) Exercise. $\square$

Let us now go back to something we mentioned at the start of Lecture 5. Suppose we wanted to define

$$\sqrt{x(x-1)(x+1)}$$

on $\mathbb{C}$. We see that we run in to the same problem as when we try to define log. The problem of course if the square root, if we wanted to define the square root as the inverse of $z \mapsto z^2$, all non zero points z will have two pre-images. If we pick one pre-image $\sqrt{z}$ for z, then walk around a path about 0, back to z, we obtain $-\sqrt{z}$. If we walk around twice, we obtain the same pre-image $\sqrt{z}$, hence in this case we only need to glue two branches together to get the correct domain of definition for $\sqrt{-}$.

We apply the same procedure to $\sqrt{x(x-1)(x+1)}$, we see that we get the same pre-image if walk around an even number of the roots $0, -1$, or 1 and the negative if we walk around an odd number. This shows that to get the correct domain of definition, we can cut out $[0, 1]$ and $(-\infty, -1]$, and glue the two branches together along these cuts. It is clear then that the corresponding space is topologically isomorphic to a torus. In fact this torus is canonically isomorphic to the space set of point $y^2 = x^3 - x$. The same procedure applis to any equation $y^2 = x^3 + ax + b$ where the cubic polynomial has the same roots.

Thus another way to think of the complex torus C/Λ_τ is the domain of definition of the function

$$\sqrt{x^3 - g_2(\tau)x - g_3(\tau)}$$